Lecture 7
Complex Analysis II

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2026-03-05

In today's lecture we continue our introduction to complex analysis by defining analytic complex functions, the complex derivative, and looking closer at rational complex functions.

Complex Derivatives

A complex function $f(s)$ is *analytic* in some domain $R \subset \mathbb{C}$ if the function is

- single valued, and
- has a finite complex derivative $f'(s) \equiv \frac{df}{ds}$ for all $s \in R$.

Let $f(s) = f(x + jy) = u(x, y) + jv(x, y)$. The function has a complex derivative if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are called the Cauchy-Riemann conditions. If these conditions are met then the complex derivative is given by

$$f'(s) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}$$

If a complex function is analytic over the entire complex plane, i.e. $R = \mathbb{C}$, it is called an *entire* function.

If a complex function is not analytic at a point s_0, but is analytic in the neighborhood of s_0, then the point is called a *singularity* or a *singular point* of the function.

Examples

Example 1: Let $s = x + jy$ and define

$$f(s) = e^s = f(x+jy) = e^{x+jy} = e^x e^{jy}$$

Using Euler's relation

$$f(x+jy) = \underbrace{e^x\cos(y)}_{u(x,y)} + j\underbrace{e^x\sin(y)}_{v(x,y)}$$

Checking the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = e^x\cos(y)$$

$$\frac{\partial u}{\partial y} = -e^x\sin(y)$$

$$\frac{\partial v}{\partial x} = e^x\sin(y)$$

$$\frac{\partial v}{\partial y} = e^x\cos(y)$$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x\cos(y)$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -e^x\sin(y)$ and the function is analytic everywhere and thus an entire function. The complex derivative is

$$f'(s) = e^x\cos(y) + je^x\sin(y) = e^x e^{jy} = e^{x+jy} = e^s$$

just as if s was real and f a real function.

Example 2: Let $f(s) = \frac{1}{s}$, for $|s| > 0$, i.e. $s \neq 0$. Then

$$f(s) = f(x+jy) = \frac{1}{x+jy} = \frac{1}{x+jy}\cdot\frac{x-jy}{x-jy} = \frac{x-jy}{x^2+y^2} = \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} + j\underbrace{\frac{-y}{x^2+y^2}}_{v(x,y)}$$

Checking the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

we we see that the function is analytic for $s \in \mathbb{C} - (0 + j0)$ and

$$f'(s) = \frac{-x^2+y^2}{(x^2+y^2)^2} + j\frac{2xy}{(x^2+y^2)^2}$$

With some work you can show this is the same as

$$f'(s) = -\frac{1}{s^2}$$

again, just as if s was real and f a real function.

Example 3: Now for a counter-example. Consider

$$f(s) = s^* = f(x+jy) = x - jy$$

Thus $u(x, y) = x$ and $v(x, y) = -y$. Checking the Cauchy-Riemann conditions,

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial y} = -1$$

we see $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$ thus the function is not analytic and no derivative exists.

Example 4: Show that $s_0 = -1$ is a singularity of an analytic complex function

$$f(s) = \frac{1}{s+1} = f(x+jy) = \frac{1}{(1+x)+jy}$$

Rearranging to get the real part and imaginary part

$$\frac{1}{(1+x)+jy}\frac{(1+x)-jy}{(1+x)-jy} = \underbrace{\frac{1+x}{(1+x)^2+y^2}}_{u(x,y)} + j\underbrace{\frac{-y}{(1+x)^2+y^2}}_{v(x,y)}$$

Checking the Cauchy-Riemann conditions,

$$\frac{\partial u}{\partial x} = \frac{-(1+x)^2+y^2}{[(1+x)^2+y^2]^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2(1+x)y}{[(1+x)^2+y^2]^2}$$

$$\frac{\partial v}{\partial x} = \frac{2(1+x)y}{[(1+x)^2+y^2]^2}$$

$$\frac{\partial v}{\partial y} = \frac{-(1+x)^2+y^2}{[(1+x)^2+y^2]^2}$$

we see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ except at $x = -1, y = 0$ where

$$\frac{-(1+x)^2+y^2}{[(1+x)^2+y^2]^2} = \frac{0}{0} = \text{undefined}$$

Thus the function is analytic except at $s_0 = -1 + j0 = -1$.

Rational Complex Functions

The previous example is a *rational function*, a ratio of polynomials in s. These will be very important later in the course. We will need to be very adept at doing manipulations of such functions. The general case can be written as

$$f(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \equiv \frac{P(s)}{Q(s)}$$

- If $M < N$ the ratio is said to be *strictly proper*.
- If $M \leq N$ the ratio is *proper*.
- If $M > N$ the ratio is *improper*.

Example: $M = 2, N = 3$

$$f(s) = \frac{b_0 + b_1 s + b_2 s^2}{a_0 + a_1 s + a_2 s^2 + a_3 s^3}$$

is a strictly proper rational function.

The roots of the denominator polynomial $Q(s)$ are the singularities of f.

Partial Fraction Expansion

We will often be interested in writing rational functions as a *partial fraction expansion* of the factors of the denominator. A ratio of polynomials that is improper can be written as the sum of a polynomial and a strictly proper rational function. Example

$$f(s) = \frac{2s^3 + 9s^2 + 11s + 2}{s^2 + 4s + 3} = (2s + 1) + \frac{s - 1}{s^2 + 4s + 3}$$

Once a rational function is proper we can expand it in terms of the factors of $Q(s)$. Example

$$f(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}$$

To find A and B we can "clear" the fractions

$$\frac{A}{s + 1} + \frac{B}{s + 2} = \frac{1}{(s + 1)(s + 2)} = \frac{A(s + 2) + B(s + 1)}{(s + 1)(s + 2)}$$

which gives the equation for the numerator

$$A(s + 2) + B(s + 1) = 1 + 0s$$

which requires $A = 1$ and $B = -1$, so that

$$f(s) = \frac{1}{s + 1} + \frac{-1}{s + 2}$$

Clearing fractions can be tedious with higher order systems. A shortcut is to use the Heaviside "cover-up" method, or finding the *residues*.

Case #1: non-repeated singularities

$$f(s) = \frac{P(s)}{(s - \alpha_1)(s - \alpha_2) \cdots (s - \alpha_N)}$$

where α_i are the distinct singularities (roots of $Q(s)$). Then

$$f(s) = \frac{K_1}{s-\alpha_1} + \frac{K_1}{s-\alpha_1} + \cdots + \frac{K_N}{s-\alpha_N}$$

To find the K_i values we multiply through by that term $(s-\alpha_i)$ and evaluate the result when $s = \alpha_i$

$$K_i = (s-\alpha_i)f(s)|_{s=\alpha_i}$$

Example:

$$f(s) = \frac{s^2+2s+7}{(s+1)(s-3)(s+5)} = \frac{K_1}{s+1} + \frac{K_2}{s-3} + \frac{K_3}{s+5}$$

We note that $M = 2$ and $N = 3$ so that the function is strictly proper. To find the constants

$$K_1 = \left.\frac{s^2+2s+7}{(s-3)(s+5)}\right|_{s=-1} = -\frac{3}{8}$$

$$K_2 = \left.\frac{s^2+2s+7}{(s+1)(s+5)}\right|_{s=3} = \frac{11}{16}$$

$$K_3 = \left.\frac{s^2+2s+7}{(s+1)(s-3)}\right|_{s=-5} = \frac{11}{16}$$

This works even if the roots are complex. For example:

$$f(s) = \frac{1}{(s+1+j)(s+1-j)} = \frac{K_1}{s+1+j} + \frac{K_2}{s+1-j}$$

$$K_1 = \left.\frac{1}{s+1-j}\right|_{s=-1-j} = -\frac{1}{2j}$$

$$K_2 = \left.\frac{1}{s+1+j}\right|_{s=-1+j} = \frac{1}{2j}$$

Case #2: However we will often want to avoid working with complex roots directly. Another approach is to combine them into a quadratic term. Here is an example:

$$f(s) = \frac{s+2}{s^3+3s^2+4s+2} = \frac{s+2}{(s+1)(s+1+j)(s+1-j)} = \frac{s+2}{(s+1)(s^2+2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}$$

To find A we use the same residue method

$$A = \left. \frac{s+2}{s^2+2s+2} \right|_{s=-1} = 1$$

To find B and C we can clear fractions or use another shortcut. To find C, let $s = 0$. We get

$$\frac{A}{1} + \frac{C}{2} = 1$$

which implies that $C = 0$. To find B multiply through by s and let $s \to \infty$.

$$\frac{As}{s+1} + \frac{Bs^2 + Cs}{s^2+2s+2} = \frac{s^2+2s}{s^3+3s^2+4s+2}$$

In the first term divide top and bottom by s, in the second term by s^2, and on the right-hand side by s^3:

$$\frac{A}{1+s^{-1}} + \frac{B + Cs^{-1}}{1+2s^{-1}+2s^{-2}} = \frac{s^{-1}+2s^{-2}}{1+3s^{-1}+4s^{-2}+2s^{-3}}$$

Now let $s \to \infty$. We get

$$\frac{A}{1} + \frac{B}{1} = \frac{0}{1} = 0$$

which implies $B = -1$.

Or you can just clear the fractions and substitute for A and C.

$$As^2 + 2As + 2A + Bs^2 + Cs + Bs + C = s + 2 \tag{1}$$

$$(A+B)s^2 + (2A + B + c)s + 2A + C = s + 2 \tag{2}$$

$$\tag{3}$$

We only need one equation, so lets use $2A + B + C = 1$. Solving for B and substituting for $A.C$ gives $B = -1$ as above.

Thus the final result is

$$f(s) = \frac{1}{s+1} + \frac{-s}{s^2+2s+2}$$

Case # 3: One other complication is when we have repeated roots (a root λ, repeated $r > 1$ times).

$$f(s)\frac{P(s)}{(s-\lambda)^r\,(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_{N-r})}$$

To handle this case we expand the ratio as

$$f(s)=\frac{a_0}{(s-\lambda)^r}+\frac{a_1}{(s-\lambda)^{r-1}}+\cdots+\frac{a_{r-1}}{s-\lambda}+\frac{K_1}{s-\alpha_1}++\frac{K_2}{s-\alpha_2}+\cdots++\frac{K_{N-r}}{s-\alpha_{N-r}}$$

The K_i values are determined using the same residue method as before. The a_i values are found using a variation on the residue method:

$$a_i=\frac{1}{i!}\frac{d^i}{ds^i}[(s-\lambda)^r f(s)]\Big|_{s=\lambda}$$

That is rather complicated so lets finish up with an example. Find the partial fraction expansion of

$$f(s)=\frac{s+3}{(s+1)^2(s+2)}$$

Using the pattern above we expand the terms as

$$f(s)=\frac{K_1}{s+2}+\frac{a_0}{(s+1)^2}+\frac{a_1}{s+1}$$

then find the residues

$$K_1=\frac{s+3}{(s+1)^2}\Big|_{s=-2}=1$$

$$a_0=\frac{1}{0!}\frac{d^0}{ds^0}\left[\frac{s+3}{s+2}\right]\Big|_{s=-1}=2$$

$$a_1=\frac{1}{1!}\frac{d}{ds}\left[\frac{s+3}{s+2}\right]\Big|_{s=-1}=\frac{(s+2)(1)-(s+3)(1)}{(s+2)^2}\Big|_{s=-1}=-1$$

Thus the result is

$$f(s)=\frac{1}{s+2}+\frac{2}{(s+1)^2}+\frac{-1}{s+1}$$

As you might expect there are computational tools that do this job for you.

- Mathematica: `Apart` function
- Matlab (using symbolic toolbox) and Maxima: `partfrac` function

You can use this to check work and help on homework, but you will need to be able to do this by hand on exams.