

# Lecture 14

## Bode Plots and Linear System ID

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### From Transfer Functions to Frequency Responses

Recall if a causal system is stable then the right-most pole is in the left-hand plane and the ROC includes the imaginary axis ( $\text{Re}(s) = 0$ ). If the system is non-causal and stable then the ROC is a strip in the complex plane that includes the imaginary axis. In either case, evaluating the transfer function  $H(s)$  on the imaginary axis gives the frequency response  $H(j\omega)$ .

$$H(s)|_{s=j\omega} = H(j\omega)$$

Thus we have another way of finding the frequency response:

1. Find the transfer function;
2. Determine if the system is stable;
3. Evaluate the transfer function on the imaginary axis.

#### Example 1

Given the following block diagram, where  $K_1$  and  $K_2$  are gain parameters, determine the transfer function  $H(s)$  and the frequency response  $H(j\omega)$  if it exists.

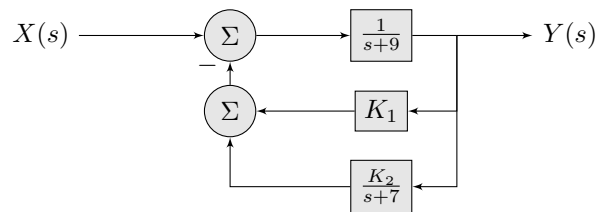


Figure 1: Block diagram for example 1.

[Step 1]

$$H(s) = \frac{s + 7}{s^2 + (16 + K_1)s + 49 + 7K_1 + K_2}$$

[Step 2]

$H(s)$  is stable if the roots of  $s^2 + (16 + K_1)s + 49 + 7K_1 + K_2$  are in the left-hand plane (in controls these are called the closed-loop poles). From the quadratic formula the roots are  $\frac{1}{2} \left( -16 - K_1 \pm \sqrt{60 + 4K_1 + K_1^2 - 4K_2} \right)$ . If the real parts are less than zero, the system is stable and the frequency response exists.

[Step 3]

$$H(j\omega) = H(s)|_{s=j\omega} = \frac{7 + j\omega}{49 + 7K_1 + K_2 - \omega^2 + j(16 + K_1)\omega}$$

## Routh-Horwitz Criteria

For larger systems the order of the denominator polynomial makes determining stability more difficult, especially when there are parameters like in the example above. A shortcut is a numerical procedure known as the Routh-Horwitz criteria.

Let  $H(s) = \frac{P(s)}{Q(s)}$ . Write  $Q(s)$  in the form

$$Q(s) = a_0s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n$$

1. If any  $a_i$  are zero, or negative with any other positive coefficients then the system is unstable.
2. Else, form the Routh array

$$\begin{array}{ccccccc} s^n & a_0 & & a_2 & a_4 & & \dots \\ s^{n-1} & a_1 & & a_3 & a_5 & & \dots \\ s^{n-2} & b_1 & & b_2 & b_3 & & \dots \\ s^{n-3} & c_1 & & c_2 & c_3 & & \dots \\ & \vdots & \vdots & \vdots & \vdots & & \dots \\ s^2 & e_1 & & e_2 & & & \\ s^1 & f_1 & & & & & \\ s^0 & g_1 & & & & & \end{array}$$

where

$$b_1 = \frac{a_1 \cdot a_2 - a_0 \cdot a_3}{a_1}$$

$$b_2 = \frac{a_1 \cdot a_4 - a_0 \cdot a_5}{a_1}$$

etc.

$$c_1 = \frac{b_1 \cdot a_3 - a_1 \cdot b_2}{b_1}$$

$$c_2 = \frac{b_1 \cdot a_5 - a_1 \cdot b_3}{b_1}$$

etc.

3. All signs of the first column ( $a_0, a_1, b_1, c_1, \dots, e_1, f_1, g_1$ ) must be positive for the system to be stable.

### Example 1 Revisited

Applying this procedure to our previous example, the Routh array is

$$\begin{array}{r} s^2 \quad 1 \qquad \qquad \qquad 49 + 7K_1 + K_2 \\ s^1 \quad 16 + K_1 \qquad \qquad \qquad 0 \\ s^0 \quad b_1 \end{array}$$

where

$$b_1 = \frac{(16 + K_1)(49 + 7K_1 + K_2) - 0(1)}{16 + K_1} = 49 + 7K_1 + K_2$$

Thus to be stable  $16 + K_1 > 0$  and  $49 + 7K_1 + K_2 > 0$ , or  $K_1 > -16$  and  $K_2 > -49 - 7K_1$ .

### Example 2

Given

$$H(s) = \frac{5s^2 + 7s + 1}{s^4 + 2s^3 + 3s^2 + 4s + 5}$$

Find the frequency response if it exists, or state why it does not exist.

$$Q(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$$

All coefficients are positive. Forming the Routh array

$$\begin{array}{rcccc}
 s^4 & 1 & & 3 & 5 & \dots \\
 s^3 & 2 & & 4 & 0 & \dots \\
 s^2 & b_1 & & b_2 & & \dots \\
 s^1 & c_1 & & & & \\
 s^0 & d_1 & & & & 
 \end{array}$$

where

$$b_1 = \frac{2 \cdot 3 - 1 \cdot 4}{2} = 1$$

$$b_2 = \frac{2 \cdot 5 - 1 \cdot 0}{2} = 5$$

$$c_1 = \frac{4 \cdot b_1 - 2 \cdot b_2}{b_1} = -6 \text{ (sign change detected)}$$

$$d_1 = \frac{c_1 \cdot b_2 - b_1 \cdot 0}{c_1} = 5 \text{ (another sign change detected)}$$

Therefore the system is unstable and  $H(j\omega)$  does not exist.

Note: if you are only interested in the stability you can stop computing the coefficients of the Routh array as soon as you detect a sign change in the first column. However, if you continue to the end, the number of sign changes tells you how many poles are in the right-hand side of the complex plane.

### Example 3

$$H(s) = \frac{-7s + 4}{s^3 + 3s^2 + 2s + 5}$$

$$Q(s) = s^3 + 3s^2 + 2s + 5$$

All the coefficients are positive. Forming the Routh array

$$\begin{array}{rcl}
s^3 & 1 & 2 \quad 0 \quad \dots \\
s^2 & 3 & 5 \quad \dots \\
s^1 & b_1 & \\
s^0 & c_1 &
\end{array}$$

where

$$b_1 = \frac{2 \cdot 3 - 1 \cdot 5}{3} = \frac{1}{3}$$

$$c_1 = \frac{5 \cdot b_1 - 3 \cdot 0}{b_1} = 5$$

No sign changes implies the system is stable. Performing the substitution the frequency response is

$$H(j\omega) = \frac{4 - j7\omega}{5 - 3\omega^2 + j(2\omega - \omega^3)}$$

## Bode Plots and System Identification

Recall the most common way to visualize a stable CT LTI system is via a Bode plot.

- semilog plot of  $|H(j\omega)|$  or  $|H(j2\pi f)|$  in dB
- semilog plot of  $\angle H(j\omega)$  or  $\angle H(j2\pi f)$  in radians or degrees

This is easy to do given an expression (with no unknown parameters) for  $H(j\omega)$  using a computer. But what if we are not given an expression? How do we proceed?

### System Identification

We may have a model for some parts of a system (blocks) but not others. When we know the inputs and outputs of a system (from experiments), but do not have a model of the system itself, we have a system identification problem. These come in a gradation, from black box (opaque) models to grey box models.

In a black box model we know very little about the system. Given the focus of the current course, we will assume at least it is a linear system, stable, and without delays. We can then conjecture it has a generic frequency response.

$$H(j\omega) = \frac{K(j\omega + b_1)(j\omega + b_2) \cdots (j\omega + b_M)}{(j\omega + a_1)(j\omega + a_2) \cdots (j\omega + a_N)}$$

Beyond that we would need to determine the order of the system  $N$ , the location of the poles  $a_i$ , and the number and location of any zeros  $M, b_i$ , as well as the overall gain  $K$ .

In a grey box model we might assume the order and number of zeros are known and only need determine the locations of any poles and zeros. For example

$$H(j\omega) = \frac{K}{(j\omega + a_1)(j\omega + a_2)}$$

A full coverage of linear system identification is outside the course scope, however we can easily measure points of  $H(j\omega)$  using sinusoidal inputs. i.e. perform a frequency sweep as illustrated below.

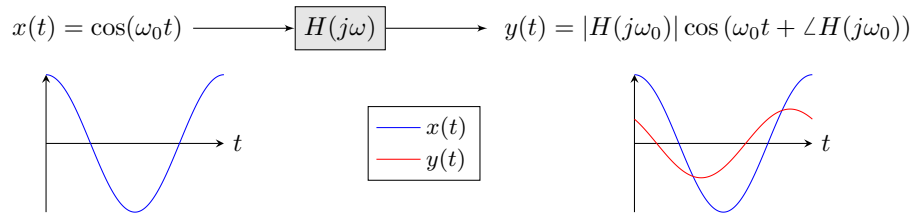


Figure 2: Illustration of linear system identification via a frequency sweep. An input sinusoid at a given frequency  $\omega_0$  is input to the system and the output sinusoid measured. Since the system is linear the input and output frequency are the same. The change in magnitude between input and output determines  $|H(j\omega_0)|$ . The time shift between input and output determines  $\frac{\angle H(j\omega_0)}{\omega_0}$ . By changing the frequency and measuring the corresponding gain and phase we can sweep out a discrete set of points on the Bode plot.

Given the frequency sweep data, knowledge of manually constructing Bode plots can give information about the order of the system, type of a filter, and a credible transfer function.

Consider a generic transfer function of a stable system

$$H(s) = \frac{(s + b_1)(s + b_2) \cdots (s + b_M)}{(s + a_1)(s + a_2) \cdots (s + a_N)}$$

where  $\text{Re}(s) > \max_k \text{Re}(a_k) < 0$ ,  $N > M$ , and  $-b_k$  are the zeros,  $-a_k$  the poles.

Rewrite by factoring as

$$H(s) = K \frac{\left(1 + \frac{s}{b_1}\right) \left(1 + \frac{s}{b_2}\right) \cdots \left(1 + \frac{s}{b_M}\right)}{\left(1 + \frac{s}{a_1}\right) \left(1 + \frac{s}{a_2}\right) \cdots \left(1 + \frac{s}{a_N}\right)}$$

where

$$K = \frac{b_1 \cdot b_2 \cdot b_3 \cdots b_M}{a_1 \cdot a_2 \cdot a_3 \cdots a_N}$$

Since it is stable let  $s = j\omega$ .

$$H(j\omega) = K \frac{\left(1 + \frac{j\omega}{b_1}\right) \left(1 + \frac{j\omega}{b_2}\right) \cdots \left(1 + \frac{j\omega}{b_M}\right)}{\left(1 + \frac{j\omega}{a_1}\right) \left(1 + \frac{j\omega}{a_2}\right) \cdots \left(1 + \frac{j\omega}{a_N}\right)}$$

Taking the magnitude

$$|H(j\omega)| = |K| \frac{\left|1 + \frac{j\omega}{b_1}\right| \left|1 + \frac{j\omega}{b_2}\right| \cdots \left|1 + \frac{j\omega}{b_M}\right|}{\left|1 + \frac{j\omega}{a_1}\right| \left|1 + \frac{j\omega}{a_2}\right| \cdots \left|1 + \frac{j\omega}{a_N}\right|}$$

Now, note the following properties of the log function

- $\log(a^b) = b \cdot \log(a)$
- $\log(a \cdot b) = \log(a) + \log(b)$
- $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$

Then when computing the magnitude in dB

$$\begin{aligned} 20 \log_{10} |H(j\omega)| &= 20 \log |K| \\ &+ 20 \log \left|1 + \frac{j\omega}{b_1}\right| + \cdots + 20 \log \left|1 + \frac{j\omega}{b_M}\right| \\ &+ 20 \log \left|1 + \frac{j\omega}{a_1}\right| + \cdots + 20 \log \left|1 + \frac{j\omega}{a_N}\right| \end{aligned}$$

We see the magnitude plot in dB is a sum of plots for each term, with the non-gain terms having an identical form. Lets look at an asymptotic approximation to  $\left|1 + \frac{j\omega}{a}\right|$  for some  $a \in \mathbb{R}$  and convert to dB

- when  $\omega \ll |a|$ ,  $20 \log \left|1 + \frac{j\omega}{a}\right| \approx |1| = 0$  dB
- when  $\omega \gg |a|$ ,  $20 \log \left|1 + \frac{j\omega}{a}\right| \approx |\omega| = 20$  dB per decade slope

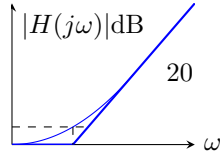


Figure 3: Asymptotic plot for first-order zero.

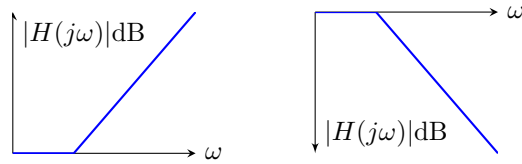


Figure 4: Asymptotic plot for first-order zero (left) and first-order pole (right).

Since the plots add, if we have a mixture of real zeros and poles the plots add or subtract accordingly with slopes adding or subtracting.

Thus the slope can be used to estimate the system order.

Cases that are not addressed above:

- zero at the origin

$$20 \log_{10} |j\omega| = 20 \log_{10}(\omega)$$

The linear approximation is equivalent to a zero approaching the  $\omega = 0$  axis.

- pole at the origin

$$20 \log_{10} \left| \frac{1}{j\omega} \right| = -20 \log_{10}(\omega)$$

The linear approximation is equivalent to a pole approaching the  $\omega = 0$  axis.

- second order with complex poles  $-a, -a^*$

$$H(s) = \frac{|a|^2}{(s+a)(s+a^*)} = \frac{\omega_0^2}{s^2 + 2\eta\omega_0 s + \omega_0^2}$$

- $\omega_0$  is the natural frequency,  $\omega_0^2 = |a|^2$
- $\eta$  is the damping ratio ( $\eta > 1$  overdamped,  $\eta < 1$  underdamped,  $\eta = 1$  critically damped),  $2\eta\omega_0 = \text{Re}(a)$
- Q factor  $Q = \frac{1}{2\eta}$

$$20 \log_{10} |H(j\omega)| = -10 \log_{10} \left\{ \left[ 1 - \left( \frac{\omega}{\omega_0} \right)^2 \right]^2 + \left[ 2\eta \left( \frac{\omega}{\omega_0} \right) \right]^2 \right\}$$

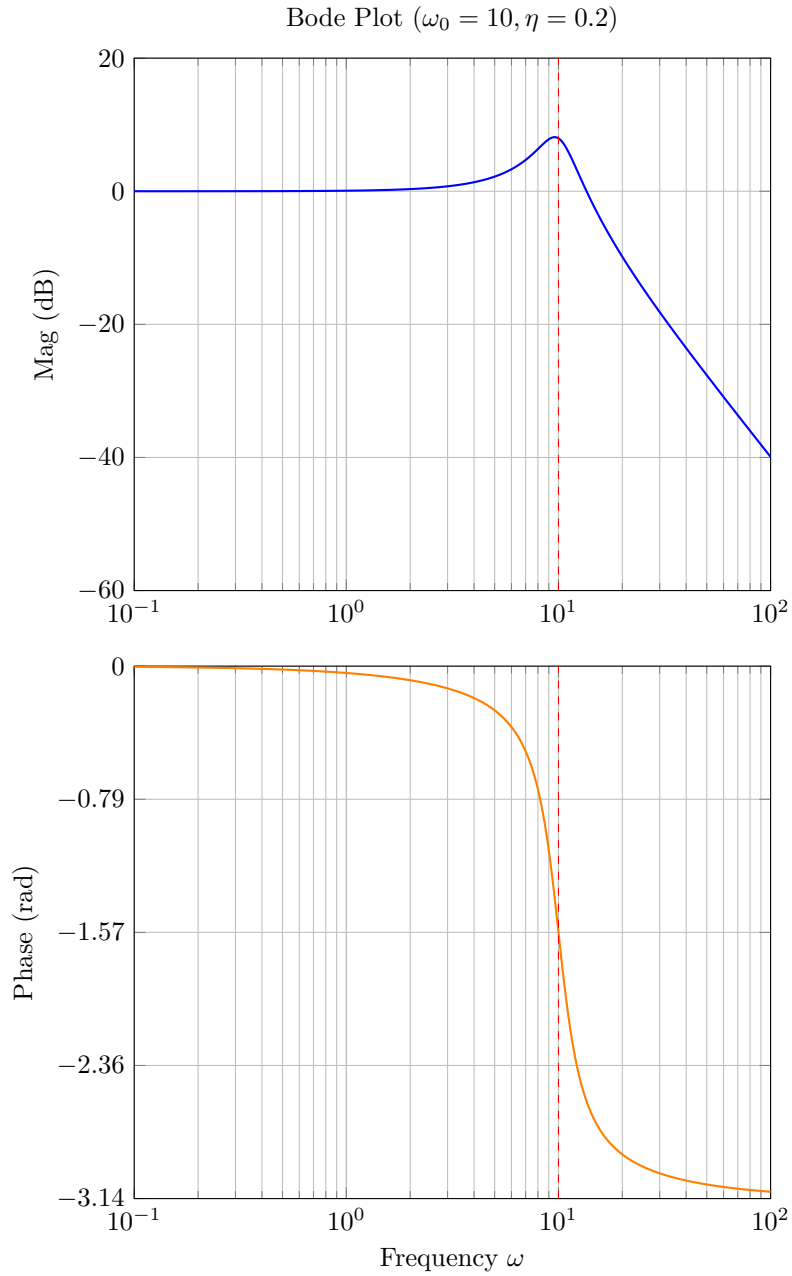


Figure 5: Illustration of a second-order system with  $\omega_0 = 10$  and  $\eta = 0.2$ . The linear approximation is a constant of zero dB up to  $\omega_0$ , followed by a linear decrease of 20 dB per decade.