Lecture 10
Inverse Laplace Transform

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In today's lecture we introduce the Inverse Laplace Transform for CT signals and systems.

Inverse Unilateral Laplace Transform

Consider a causal function $f(t)$ that may or may not be absolutely integrable. Let $g(t) = f(t)e^{-ct}$ for some $c > 0$ such that

$$\int_0^\infty |g(t)|dt = \int_0^\infty \left|f(t)e^{-ct}\right| dt < \infty$$

Then the Fourier Transform of $g(t)$ exists

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt = \int_0^\infty f(t)e^{-ct}e^{-j\omega t}dt \\ &= \int_0^\infty f(t)e^{-(c+j\omega)t}dt \\ &= \mathcal{L}_1\{f(t)\}\big|_{s=c+j\omega} \quad c \text{ fixed s.t. } c + j\omega \in \text{ROC} \end{aligned}$$

The inverse Fourier Transform is

$$g(t) = f(t)e^{-ct} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega$$

multiply through by e^{ct}

$$
\begin{aligned}
e^{ct}g(t) = f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{ct}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{(c+j\omega)t}d\omega
\end{aligned}
$$

Substitute $G(\omega)$

$$
f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\underbrace{\int_{0}^{\infty} f(\tau)e^{-(c+j\omega)\tau}\,d\tau}_{F(s)|_{s=c+j\omega}}\right]e^{(c+j\omega)t}d\omega
$$

$$
f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(c+j\omega)e^{(c+j\omega)t}d\omega
$$

Let $s = c + j\omega$. Since c is a constant $ds = 0 + jd\omega$ implying $d\omega = \frac{1}{j}ds$ and

$$
f(t) = \frac{1}{2\pi j}\int\limits_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds
$$

a complex integral.

Note: we can think of c as the real values such that $f(t)e^{-ct}$ has a Fourier Transform. This is the ROC.

Does the above integral look like our contar Integrals from previous lectures?

Yes, if we use the Bromwitch contour.

$$
f(t) = \frac{1}{2\pi j}\int\limits_{c-j\infty}^{c+j\infty} F(s)e^{st}ds = \lim_{R\to\infty}\oint_{C(R)} F(s)e^{st}ds
$$

where $C(R)$ is the Bromwitch contour.
This allows us to use the method of residues since $C \in$ ROC implies $C(R)$ encloses all singulartes of $F(s)$.

Example: Recall that $\mathcal{L}\{x(t)\} = \mathcal{L}\left\{e^{-at}u(t)\right\}$ for $a \in \mathbb{R}$ was $\frac{1}{s+a}\mathrm{Re}(s) > -a$

Then

$$
x(t) = \mathcal{L}_1\left\{\frac{1}{s+a}\right\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{1}{s+a}e^{st}ds \quad \text{for} c > -a
$$

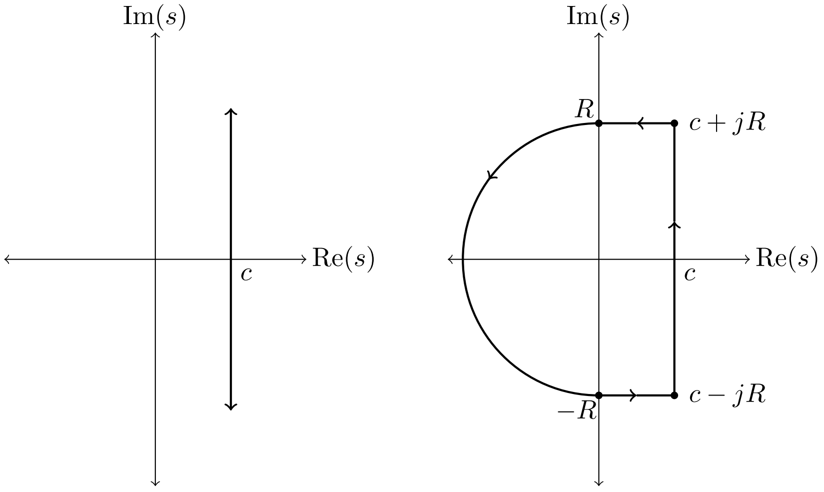

Figure 1: The linear contour of integration for the inverse Laplace transform becomes the Bromwitch contour as $R \to \infty$.

or using the Bromwitch contour

$$x(t) = \lim_{R\to\infty} \frac{1}{2\pi j} \oint_{C(R)} \frac{e^{st}}{s+a} ds \qquad c > -a$$

Now from last time, since e^{st} is analytic and $C(R)$ encloses the singulary at $-a$, from the Residue Theorem

$$\oint \frac{e^{st}}{s+a}\, ds = 2\pi j k_1$$

where the residual is

$$k_1 = (s+a)\frac{e^{st}}{s+a}\bigg|_{s=-a} = e^{st}\big|_{s=-a} = e^{-at} \qquad t > 0$$

Thus

$$x(t) = \frac{1}{2\pi j} \cdot 2\pi j e^{-at} \qquad t > 0 = e^{-at}u(t)$$

Inverse Bilateral Laplace Transform

Given a non-causal signal $x(t)$ we can write it as

$$x(t) = \underbrace{x_1(t)u(-t)}_{\text{anticausal part}} + \underbrace{x_2(t)u(t)}_{\text{causal part}}$$

$$\begin{aligned}
L_2\left\{x(t)\right\} &= \int_{-\infty}^{0} x_1(t)e^{-st}\,dt + \int_{0}^{\infty} x_2(t)e^{-st}\,dt \\
&= \int_{0}^{\infty} x_1(-t)e^{st}\,dt + \mathcal{L}_1\left\{x_2(t)\right\} \\
&= \mathcal{L}_1\left\{x_1(-t)\right\}\big|_{s=-s} + \mathcal{L}_1\left\{x_2(t)\right\} \\
X(s) &= \underbrace{X_1(s)}_{\mathrm{Re}(s)<U} + \underbrace{X_2(s)}_{\mathrm{Re}(s)>L}
\end{aligned}$$

where the ROC of $X(s)$ is the intersection of the ROC for X_1 and X_2, a strip in the complex plane $L < \mathrm{Re}(s) < U$.

To use this for the inverse we apply this in reverse.

$$x(t) = \mathcal{L}_1^{-1}\left\{X_1(-s)\right\}\big|_{t\to -t} + \mathcal{L}_1^{-1}\left\{X_2(s)\right\}$$

Example: Recall an example of a forward bilateral Laplace transform from lecture 9

$$\mathcal{L}_2\left\{e^{-|t|}\right\} = \underbrace{\frac{-1}{s-1}}_{\mathrm{Re}(s)<1} + \underbrace{\frac{1}{s+1}}_{\mathrm{Re}(s)>-1} = X(s)$$

Then

$$x(t) = \mathcal{L}_2^{-1}\{X(s)\} = \mathcal{L}_1^{-1}\left\{\frac{1}{s+1}\right\}\Bigg|_{t\to -t} \mathcal{L}_1^{-1}\left\{\frac{1}{s+1}\right\}$$

From our previous result when $a = 1$

$$\begin{aligned}
x(t) &= e^{-t}u(t)\big|_{t\to -t} + e^{-t}u(t) \\
&= e^{t}u(-t) + e^{-t}u(t) = e^{-|t|}
\end{aligned}$$

Example: Find the inverse Laplace transform of

$$X(s) = \frac{1}{s} \quad \mathrm{Re}(s) > 0\ .$$

Solution: Since the ROC corresponds to a causal signal, for $t > 0$

$$\begin{aligned}
x(t) &= \frac{1}{2\pi j} \int\limits_{c-j\infty}^{c+j\infty} X(s) e^{st}\, ds \quad c > 0 \\
&= \frac{1}{2\pi j} \oint \frac{e^{st}}{s}\, ds \\
&= \frac{1}{2\pi j} 2\pi j\ k \\
&= s \frac{e^{st}}{s} \bigg|_{s=0} \\
&= 1 \quad t > 0
\end{aligned}$$

which can be written as $x(t) = u(t)$ for all times t.

Example: Find the inverse Laplace transform of

$$X(s) = \frac{10}{s^2 + 5s + 6} \quad \mathrm{Re}(s) > -2\ .$$

First we use the partial fraction expansion.

$$X(s) = \frac{10}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$A = \frac{10}{s+3}\bigg|_{s=-2} = \frac{10}{1} = 10$$

$$B = \frac{10}{s+2}\bigg|_{s=-3} = \frac{10}{-1} = -10$$

Then

$$x(t) = \mathcal{L}_1^{-1}\left\{\frac{10}{s+2}\right\} + \mathcal{L}_1^{-1}\left\{\frac{-10}{s+3}\right\}$$

Using the Residue theorem,

$$
\begin{aligned}
x(t) &= \frac{10}{2\pi j}\int\limits_{c-j\infty}^{c+j\infty}\frac{e^{st}}{s+2}\,ds - \frac{10}{2\pi j}\int\limits_{c-j\infty}^{c+j\infty}\frac{e^{st}}{s+3}\,ds \\
&c>-2 \quad c>-2 \\
&= \frac{10}{2\pi j}\oint\frac{e^{st}}{s+2}\,ds - \frac{10}{2\pi j}\oint\frac{e^{st}}{s+3}\,ds \\
&= \frac{10}{2\pi j}2\pi j e^{-2t} - \frac{10}{2\pi j}2\pi j e^{-3t} \\
&= 10e^{-2t} - 10e^{-3t} \quad t>0
\end{aligned}
$$

Writing the expression for all t gives:

$$x(t) = 10\left(e^{-2t} - e^{-3t}\right)u(t).$$

Example: Here is an example with complex singularties. Find the inverse Laplace transform of

$$X(s) = \frac{s}{s^2+2s+5} \quad \mathrm{Re}(s) > -1$$

Using the partial fraction expansion

$$X(s) = \frac{s}{(s+1+j2)(s+1-j2)} = \frac{A}{s+1+j2} + \frac{B}{s+1-j2}$$

where

$$A = \frac{-1-j2}{-1-j2+1-j2} = \frac{-1-j2}{-j4} = \frac{1}{2} - \frac{1}{4}j$$

$$B = \frac{-1+j2}{-1+j2+1+j2} = \frac{-1+j2}{j4} = \frac{1}{2} + \frac{1}{4}j$$

Then

$$x(t) = \mathcal{L}_1^{-1}\left\{\frac{\frac{1}{2}-\frac{1}{4}j}{s+1+j2}\right\} + \mathcal{L}_1^{-1}\left\{\frac{\frac{1}{2}+\frac{1}{4}j}{s+1-j2}\right\}$$

Using residues, let $c = 0 > -1$

$$x(t) = \frac{1}{2\pi j}\int\limits_{-j\infty}^{j\infty}\frac{\frac{1}{2}-\frac{1}{4}j}{s+1+j2}e^{st}\,ds + \frac{1}{2\pi j}\int\limits_{-j\infty}^{j\infty}\frac{\frac{1}{2}+\frac{1}{4}j}{s+1-j2}e^{st}\,ds$$

$$x(t) = \frac{1}{2\pi j} 2\pi j \, k_1 + \frac{1}{2\pi j} 2\pi j \, k_2$$

$$\begin{aligned}
k_1 &= \left(\frac{1}{2} - \frac{1}{4}j\right) e^{(-1-j2)t} \quad k_2 = \left(\frac{1}{2} + \frac{1}{4}j\right) e^{(-1+j2)t} \\
x(t) &= k_1 + k_2 = \left(\frac{1}{2} - \frac{1}{4}j\right) e^{(-1-j2)t} + \left(\frac{1}{2} + \frac{1}{4}j\right) e^{(-1+j2)t} \\
&\text{for } t > 0
\end{aligned}$$

with some effort you can write this as

$$x(t) = e^{-t} \left(\cos(2t) - \frac{1}{2}\sin(2t)\right) \, u(t)$$

for all t.